

A FILTRATION OF UNORIENTED COBORDISM

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ABSTRACT. We give a filtration of the unoriented cobordism ring using the infinite symplectic group, with polynomial generators given one at a time. The generating manifolds are also constructed using the cup construction.

In this paper we give a homotopy filtration of the unoriented cobordism MO , with the polynomial generators constructed and given one at a time. Such filtration arises from the corresponding filtration given for BO . The filtration is indexed on triples of integers (n, j, i) with $n \geq 1$ and $i, j \geq 0$ (with the exception that when $n = 1, j \geq 1$). We order such triples in the following way: $(n, j, i) < (n', j', i')$ when

$$\begin{array}{ll} n < n', & \text{or} \\ n = n' \text{ and } j < j', & \text{or} \\ n = n', j = j' \text{ and } i < i'. \end{array}$$

Our main theorem is the following. Mod 2 coefficients are assumed throughout.

Theorem. *There exist a filtration $F_{(n,j,i)}$ of BO such that if $MF_{(n,j,i)}$ is the Thom complex of the inclusion $F_{(n,j,i)} \rightarrow BO$, then $\pi_*(MF_{(n,j,i)})/\pi_*(MF_{<(n,j,i)})$ is a polynomial algebra $\mathbb{Z}/2\mathbb{Z}[x]$ on one generator.*

Recall [5] that the unoriented cobordism ring $\pi_*(MO) = \mathbb{Z}/2\mathbb{Z}[x_2, x_4, x_5, \dots]$ is a polynomial algebra with one generator x_n in degree n for each n not of the form $2^i - 1$ for some $i > 0$. It is well known that $\mathbb{R}P^{2n}$ can be taken to be the polynomial generator of degree $2n$ of the cobordism ring. In [2], Dold constructs manifolds that represent the odd dimensional polynomial generators of MO_* .

We are interested in mapping certain H-spaces into BO , where the homology of the associated Thom complexes include the dual Steenrod algebra \mathcal{A}_* as a tensor product. The key here is that the Adams spectral sequence collapses for such spaces. One such space is $\Omega^2 S^3$. Recall that

$$H_*(\Omega^2 S^3) \cong \mathbb{Z}/2\mathbb{Z}[\xi_1, \xi_2, \dots] \cong \mathcal{A}_*$$

where $|\xi_i| = 2^i - 1$. It is an observation of Mahowald that there exist a map $\Omega^2 S^3 \rightarrow BO$ whose Thom complex is the Eilenberg-Mac Lane spectrum $K(\mathbb{Z}/2\mathbb{Z})$. Let us make this point precise.

Let $\eta : S^1 \rightarrow BO \in \pi_1(BO)$ be the generator. Since BO is a double loop space, $BO \simeq \Omega^2 X$ for some X . Let γ be the composite

$$\Omega^2 S^3 = \Omega^2 \Sigma^2 S^1 \xrightarrow{\Omega^2 \Sigma^2 \eta} \Omega^2 \Sigma^2 BO \simeq \Omega^2 \Sigma^2 \Omega^2 X \rightarrow \Omega^2 X \simeq BO$$

Mahowald has shown in [4] that if $M(\gamma)$ is the Thom complex associated with γ , then $M(\gamma) \cong K(\mathbb{Z}/2\mathbb{Z})$.

Recall that $\mathbb{Z} \times BSp$ appears as the fourth space in the real K -theory spectrum KO , that is $\Omega^4 BSp \simeq BO \times \mathbb{Z}$, and so $\Omega^3 Sp \simeq BO \times \mathbb{Z}$. Here $Sp = \varinjlim Sp(n)$ is the

Date: November 28, 2011.

infinite symplectic group. We will show how the homology of Thom complexes of $\Omega^3 Sp(n)$ includes \mathcal{A}_* as a tensor product, and thus we are lead to considering the maps $\Omega^3 Sp(n) \rightarrow \text{BO}$, and in fact this is where the filtration in the above theorem arises.

Let us start by recalling some classical definitions and results.

Let A be an algebra over a ring R . Then elements $x_1, x_2, \dots, x_n, \dots$ in A are said to form a *simple system of generators* for A if the monomials $x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_m^{\epsilon_m}$ ($\epsilon_i = 0$ or 1 , $m \geq 0$) form a basis for A over R . For example in an exterior algebra $E(x_1, x_2, \dots)$, the elements x_1, x_2, \dots form a simple system of generators. In the polynomial ring $R[x]$, the monomials $x, x^2, x^4, \dots, x^{2^n}, \dots$ form a simple system of generators.

For a path fibration $\Omega B \rightarrow PB \rightarrow B$ over a space B , let $\sigma' : \tilde{H}_{n-1}(\Omega B) \rightarrow \tilde{H}_n(B)$ be the homology suspension.

Theorem. (*A. Borel*) *Let B be simply connected H -group. Let a_1, a_2, \dots element of $H_*(\Omega B, R)$ such that for each n only finitely many a_i 's lie in $H_*(\Omega B, R)$ and $\sigma'(a_1), \sigma'(a_1) \dots$ form a simple system of generators for the Pontrjagin ring $H_*(B, R)$. Then $H_*(\Omega B, R) \cong R[a_1, a_2, \dots]$.*

Proof. See [6], theorem 15.60. □

Using the fibration $Sp(n-1) \rightarrow Sp(n) \rightarrow S^{4n-1}$, one can easily see that:

Lemma. *The homology of $Sp(n)$ is an exterior algebra on generators in degrees of the form $4i-1$ for $1 \leq i \leq n$, that is*

$$H_*(Sp(n)) = E(a_{4i-1}, 1 \leq i \leq n).$$

For details see [3], corollary 4D.3.

Borel's theorem then implies that the homology of the loop space $\Omega Sp(n)$ is a polynomial algebra on generators in degrees $4i-2$:

$$H_*(\Omega Sp(n)) = \mathbb{Z}/2\mathbb{Z}[b_{4i-2}, 1 \leq i \leq n]$$

For the homology of the double loop space $\Omega^2 Sp(n)$, we choose a simple system of generators for the above polynomial algebra (i.e. $b_{4i-2}^{2^j}$ for $1 \leq i \leq n$ and $j \geq 0$) and $H_*(\Omega^2 Sp(n))$ will then be a polynomial algebra on generators

$$\{\sigma'(b_{4i-1}), \sigma'(b_{4i-1}^2), \sigma'(b_{4i-1}^4), \dots, 1 \leq i \leq n\}$$

We are going to consider the Thom complexes of the maps $\Omega^3(Sp(n)) \rightarrow \text{BO}$. For each n we obtain an infinite family of polynomial generators given by such complexes. We start with $n = 1$.

Sp(1). It is easily seen that $Sp(1) = S^3$, and so we are looking at the map $\Omega^3 S^3 \rightarrow \text{BO}$. The homology of ΩS^3 is a polynomial algebra on one generator in degree 2 and ΩS^3 has a CW structure

$$\Omega S^3 = J(S^2) \simeq S^2 \cup e^4 \cup e^6 \cup \dots$$

with all the cells attached nontrivially. Here $J(S^2)$ denotes the James reduced product on S^2 . Now we consider pieces of this product starting with $\Omega^2(S^2 \cup e^4 \cup e^6)$. Let \doteq denote homological equivalence. Since the 6-cell e^6 is the product of the cells e^2 and e^4 , we have

$$\Omega^2(S^2 \cup e^4 \cup e^6) \doteq \Omega^2 S^2 \times \Omega^2 S^4$$

The homology of $\Omega^2 S^2$ is \mathbb{Z} in degree zero and is equal to the homology of $\Omega^2 S^3$ in higher dimensions (this can easily be seen from the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$). In fact, $\Omega^2 S^3$ is homotopy equivalent to the connected component of $\Omega^2 S^2$ containing the base point. It is worth noting that the map γ introduced above is the composite map

$$\Omega^2 S^3 \hookrightarrow \Omega^2 S^2 \rightarrow \Omega^3 S^3 \rightarrow \text{BO}$$

and the 0^{th} space in our filtration $F_{(1,0,0)}$ is the image of this map in BO .

For $\Omega^2 S^4$, we can write

$$\Omega^2 S^4 \simeq \Omega \Omega \Sigma S^3 \simeq \Omega J(S^3)$$

We are now going to consider the pieces $J_{2^i-1}(S^3)$, for various i , starting with $J_0(S^3) = S^3$. Let γ_2 be the map $\Omega^2 S^2 \times \Omega J_0(S^3) \rightarrow \text{BO}$, and let $M(\gamma_2)$ be the associated Thom complex. The homology of this complex is given by

$$H_*(M(\gamma_2)) \cong \mathcal{A}_* \otimes H_*(\Omega S^3)$$

the best possible outcome, for feeding this into the Adams spectral sequence we get

$$\begin{aligned} E_2^{s,t} &= \text{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{Z}/2\mathbb{Z}, \mathcal{A}_* \otimes H_*(\Omega S^3)) = \text{Ext}_{\mathbb{Z}/2\mathbb{Z}}^{s,t}(\mathbb{Z}/2\mathbb{Z}, H_*(\Omega S^3)) \\ &= \begin{cases} \text{Hom}_{\mathbb{Z}/2\mathbb{Z}}^t(\mathbb{Z}/2\mathbb{Z}, H_*(\Omega S^3)) & s = 0 \\ 0 & s > 0 \end{cases} = \begin{cases} H_t(\Omega S^3) & s = 0 \\ 0 & s > 0 \end{cases} \end{aligned}$$

Hence

$$\pi_*(M(\gamma_2)) \cong H_*(\Omega S^3) = \mathbb{Z}/2\mathbb{Z}[x_2]$$

Thus we obtain our first polynomial generator x_2 in degree 2.

Next we consider the Thom complex $M(\gamma_5)$ of the map

$$\gamma_5 : \Omega^2 S^2 \times \Omega J_1(S^3) \rightarrow \text{BO}$$

The argument above repeats to give that

$$\pi_*(M(\gamma_5)) \cong H_*(\Omega J_1(S^3)) \cong \mathbb{Z}/2\mathbb{Z}[x_2, x_5]$$

giving rise to a new generator in degree $2 \cdot 3 - 1 = 5$ denoted by x_5 .

We can repeat this process by forming the Thom complexes of the maps

$$\gamma_{2^i-1} : \Omega^2 S^2 \times \Omega J_{2^i-1}(S^3) \rightarrow \text{BO}$$

giving rise to a family of generators in degrees $3 \cdot 2^i - 1$, for all nonnegative i . In addition, we take $F_{(1,1,i)}$ to be the image of γ_{2^i-1} in BO .

We now construct manifold generators representing this family. Let X be any space. Following [1], let *cup "m" construction* $P(m, X)$ on X be the quotient of $S^m \times X \times X$ by the relations $(u, x, y) \sim (-u, y, x)$. If X is an n -manifold, then $P(m, X)$ is an $(m + 2n)$ -manifold. We use this construction for $m = 1, 2$ only. Proposition 4.1 of [1] implies that if a manifold M represent an indecomposable cobordism class, then $P(1, M)$ is also indecomposable. In addition, if M is of even dimension, then $P(2, M)$ is also indecomposable.

We construct our manifolds recursively. Let M_2 be the real projective space $\mathbb{R}P^2$, representing x_2 . By the paragraph above, $M_5 = P(1, M_2)$ is a manifold of dimension 5 which represents an indecomposable cobordism class, and can be taken to be x_5 . If $M_{3 \cdot 2^i-1}$ is constructed, we can set $M_{3 \cdot 2^{i+1}-1} = P(1, M_{3 \cdot 2^i-1})$ to get a manifold of dimension $2(3 \cdot 2^i - 1) + 1 = 3 \cdot 2^{i+1} - 1$, representing $x_{3 \cdot 2^{i+1}-1}$.

The next step is to consider the next bit of the product, that is, we consider $\Omega^2 J_{2^3-1}(S^2)$, and homologically

$$\Omega^2 J_7(S^2) \doteq \Omega^2 S^2 \times \Omega^2 S^4 \times \Omega^2 S^8$$

Similar to the above case, $\Omega^2 S^8 = \Omega \Omega \Sigma S^7 = \Omega J(S^7)$, and we can consider the maps $\Omega^2 S^2 \times \Omega^2 S^4 \times \Omega J_{2^i-1}(S^7) \rightarrow \text{BO}$ for various i . The resulting Thom complexes will give new polynomial generators in degrees $7 \cdot 2^i - 1$, in addition to the generators arising from $\Omega^2 S^2 \times \Omega^2 S^4$.

We use cup 2 and then cup 1 construction to construct manifolds representing this family of generators. The lowest degree generator is in degree 6, and so we can start with $M_6 = \mathbb{P}(2, M_2)$, and if $M_{7 \cdot 2^i-1}$ is constructed, we set $M_{7 \cdot 2^{i+1}-1} = P(1, M_{7 \cdot 2^i-1})$. The dimension of $M_{7 \cdot 2^{i+1}-1}$ is then $2(7 \cdot 2^i - 1) + 1 = 7 \cdot 2^{i+1} - 1$, and can be taken to represent $x_{7 \cdot 2^{i+1}-1}$.

It should now be clear that the piece $\Omega^2 J_{2^j-1}(S^2)$, for a fixed $j \geq 1$, will give infinitely many generators in degrees $(2^{j+1} - 1)2^i - 1$ for all nonnegative i . The lowest degree generator will be in dimension $2^{j+1} - 2$, which can be constructed from the generator in degree $2^j - 2$ by cup 2 construction starting with $\mathbb{R}P^2$. The higher degree generators in the family are then given using cup 1 construction. This shows that for $n = 1$ all the generators can be built from $\mathbb{R}P^2$ recursively.

Sp(2). The homology ring $H_*(Sp(2)) = E(a_3, a_7)$ has a generator in degree 7 and $\frac{Sp(2)}{Sp(1)} = S^7$. Again homologically we have

$$\Omega S^7 \doteq \prod_{j \geq 0} S^{6 \cdot 2^j}$$

and so

$$\Omega^3 S^7 \doteq \Omega^2 \prod S^{6 \cdot 2^j} \doteq \Omega \prod \Omega \Sigma S^{6 \cdot 2^j-1} \doteq \Omega \prod J(S^{6 \cdot 2^j-1})$$

As before, starting with S^5 , we consider the Thom complexes of the maps

$$\Omega^3 S^3 \times \Omega J_{2^i-1} S^5 \rightarrow \text{BO}$$

The homology of such complexes will include the dual Steenrod algebra \mathcal{A}_* as a tensor product, resulting from $\Omega^3 S^3$. Thus in addition to the generators obtained from $Sp(1)$, we get polynomial generators in degrees $5 \cdot 2^i - 1$, $i \geq 0$. This is the first family of generators arising from $Sp(2)$, with the first one in degree 4. The representing manifolds can then be constructed recursively, starting from $\mathbb{R}P^4$ and using cup 1 construction, analogous to the case of $Sp(1)$.

Similarly, S^{11} gives generators in degrees $11 \cdot 2^i - 1$, with the first one in degree 10. For this family, we start with $M_{10} = \mathbb{P}(2, \mathbb{R}P^4)$ and apply cup 1 construction to get the higher degree generators. Similarly for S^{23}, S^{47}, \dots . This shows that $Sp(2)$ gives rise to generators in degrees $(6 \cdot 2^j - 1)2^i - 1$ for all nonnegative i and j , all of which can be constructed from $\mathbb{R}P^4$.

Sp(n). Quite generally, assume that polynomial generators are constructed for the map $\Omega^3 Sp(n-1) \rightarrow \text{BO}$. $Sp(n)$ will then give us a new exterior generator in degree $4n - 1$, and $\frac{Sp(n)}{Sp(n-1)} = S^{4n-1}$. Homologically we have

$$\Omega S^{4n-1} \doteq \prod_{j \geq 0} S^{(4n-2)2^j}$$

and so

$$\Omega^3 S^{4n-1} \doteq \Omega^2 \prod S^{(4n-2)2^j} \doteq \Omega \prod \Omega \Sigma S^{(4n-2)2^j-1} \doteq \Omega \prod J(S^{(4n-2)2^j-1})$$

This shows that the various pieces $J_{2^i-1}(S^{(4n-2)2^j-1})$ will give rise to generators in degrees $((4n-2)2^j-1)2^i-1$ for $n \geq 1$ and all nonnegative i and j (when $n = 1$, $j \geq 1$). $F_{(n,j,i)}$ is then the image of $Sp(n-1) \times \Omega J_{2^i-1}(S^{(4n-2)2^j-1})$ under the map

$$Sp(n-1) \times \Omega J_{2^i-1}(S^{(4n-2)2^j-1}) \rightarrow \Omega^3 Sp(n) \rightarrow BO$$

Thus for fixed n and j , we obtain an infinite family of generators with the first one in degree $(4n-2)2^j-2$ for $i = 0$. Manifolds representing such family are constructed using the cup construction, in the following manner. For a fixed n , we have infinite family of generators corresponding to each $j = 0, 1, 2, \dots$. When $j = 0$, the family of generators obtained by varying i can be constructed using cup 1 construction recursively starting with $\mathbb{R}P^{4(n-1)}$. For infinite family corresponding to $j = 1$, we start with cup 2 construction on $\mathbb{R}P^{4(n-1)}$, and then cup 1 for the rest of the generators in the family. The lowest degree generator in the family corresponding to $j = 2$ is obtained from cup 2 construction on lowest degree generator in the family corresponding to $j = 1$, and then cup 1 for the rest of the family. And so on. The only exception is when $n = 1$, in which case we start with $\mathbb{R}P^2$, as demonstrated above for $Sp(1)$.

Now it's easily seen that the numbers $((4n-2)2^j-1)2^i-1$ are not of the form 2^k-1 for any k , and any integer not of the form 2^k-1 can be written uniquely in the form $((4n-2)2^j-1)2^i-1$. Thus we obtain all the polynomial generators of MO_* , all of which can be constructed using cup 1 and cup 2 constructions from $\mathbb{R}P^2$ and $\mathbb{R}P^{4n}$ for $n \geq 1$.

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